Newton's Laws of System Dynamics

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Abstract

This paper proposes an understanding of system dynamics models by analogy with Newtonian mechanics. By considering the second derivative form of a model, and extending the concept of feedback loop impact, it is shown that Newton's three laws of motion have their equivalent in system dynamics, and that the impacts of the forces on the stocks are the measure of the force that determines stock behavior. The concepts of mass, inertia, momentum, and friction are explored as to their usefulness in understanding model behavior. The Newtonian understanding is applied to two standard system dynamics models, inventory-workforce and economic long-wave, where their behavior is analyzed using force dominance on the stocks and the laws of motion. The method, and conceptual understanding, is commended for further exploration.

Key Words: System dynamics, force, Newton's laws of motion, loop impact, feedback loop, loop dominance

1 Introduction

A fundamental principle of system dynamics is that the structure of a system is responsible for its behavior (Sterman, 2000, p. 28). Such structure is expressed in the stock/flow relationships and the feedback loops between the model variables. The latter are particularly important as they represent endogeneity in the system, and help explain complex behavior in the variables through the change in dominance of different types of feedback.

In order to quantify feedback loop dominance, numerous methods have been developed (Kampmann & Oliva, 2009; Duggan & Oliva, 2013). Broadly, the methods can be divided into two categories: those that relate loop gains to system behavior, expressed in eigenvalues and eigenvectors, through elasticity analysis (e.g. Forrester, 1982; Goncalves, 2009; Kampmann, 2012); and those that relate pathway connections to variable behavior expressed in their change and graphical curvature over time (e.g. Ford, 1999; Mojtahedzadeh et.al., 2004; Hayward, 2012; Hayward &

Boswell, 2014). In each method the main structural element under consideration is the feedback loop, whose changes in dominance become the chief way of explaining variable behavior. The two categories of methods are related: the loop gain is the product of the link, or pathway, gains between adjacent elements in the loop (Kampmann, 2012); also the loop gain is the product of the impacts between adjacent stocks in the loop (Hayward & Boswell, 2014), that is, the pathway participations (Mojtahedzadeh et.al., 2004). Whatever method is used, a loop with n stocks has n degrees of freedom, and thus requires n numbers to fully describe its effect, whether they are metrics, eigenvalues, or impacts. If $n > 1$ the loop gain is insufficient on its own to capture the loop dynamics.

System dynamics is not the only modeling methodology that uses structure to explain behavior; such linkage between the expression of a model and its results is the essence of all modeling whether it is, for example, agent-based, chaos theory, or Newtonian mechanics. Where such methodologies differ from system dynamics is the varied way in which "structure" is expressed in, for example, transition diagrams, abstract mathematical relationships, or physical laws. One of the attractions of system dynamics is that structure is refined to visual expressions, causal loop diagrams and stock/flow diagrams, which cut through the model's complexity to provide an intuitive framework from which model behavior can be explained. Nevertheless it may be possible to provide additional understanding of a system dynamics model by applying a different approach to model structure. This paper explores such an approach.

The starting point for such an alternative understanding of structure is to note that the link between structure and behavior in the pathway participation metric (PPM) method (Mojtahedzadeh et.al., 2004), and the loop impact method (Hayward & Boswell, 2014), is an equation for the second derivative of stock variables in terms of other variables. In each method the model equations are differentiated, placing them in second derivative form, thus focusing on the curvature in variable behavior, whilst retaining the causal structure of the model in each term that contributes to that curvature. In essence the understanding is close to that of Newtonian mechanics where acceleration is determined by various forces, here identified with different loops. However it is clear from the loop impact method that the "forces" on a given stock are less connected with a loop, but more with the effect of adjacent stocks which may be part of a loop, or may be exogenous. Indeed the loop impact method, which may be better termed the stock impact method, has made steps towards interpreting model behavior in Newtonian mechanics terms, even if this was not explicitly made clear (Hayward & Boswell, 2014).

The purpose of this present paper is to use concepts from Newtonian mechanics to understand system dynamics models[1]. Firstly, the concept of loop impact, as introduced by Hayward & Boswell (2014), is investigated further by discussing the notion of the "impact" of a cause on motion generally. The concept is reinterpreted in as the impact of a force on a stock. Secondly, laws of system dynamics are proposed by analogy with Newton's laws of motion, with the concepts of force, momentum, mass, inertia and friction explored as to their usefulness in understanding the behavior of any system dynamics model. Finally the ideas presented are applied to two existing system dynamics models to evaluate their use.

Put informally this paper addresses the question: If Sir Isaac Newton had tried to understand a system dynamics model, without the benefit of Prof. Jay Forrester's insight, how would he have explained its behaviour?

[1] Others have used system dynamics to illustrate Newtonian mechanics, e.g. Creative Learning Exchange (n.d.). However this paper attempts the opposite: to understand system dynamics models and behavior in Newtonian terms.

2 The Concept of "Impact"

Initial Considerations

The proposal is to define the concept of the impact of the force exerted by one stock on the motion of another stock, in particular its acceleration. In order to help understand the notion of impact on motion, an example is given of how the word "impact" is used with regard to motion generally.

The following situation happened to the author at a recent International System Dynamics Conference. For readability the narrative is given in the first person.

> *During the conference lunch break I wished to attend the presentation by one of the software developers. Realizing I was late for the meeting I rushed to the lecture theater, which turned out to be a massive tiered room, seating hundreds. The presentation had already started and the presenter was explaining his wares to the solitary person in the audience, who sat somewhere in the middle of the room. My entrance was greeted with a "Wow! A hundred percent increase. Do come in". It is clear my entrance had impact on the relieved presenter. I took a seat, and the presentation carried on.*
>
> *Two minutes later another person arrived. The presenter again noted his appearance and asked him to sit down, but he said it with a little less emotion than my entrance. Two minutes later yet another person arrived, now an audience of three became four. The presenter's comments were briefer still. A new arrival continued roughly every two minutes until we were up to seven people, by which stage the presenter had stopped noting their appearance. It struck me at the time that although the addition in numbers in the room was uniform, the impact of the change on the presenter became smaller the larger the audience became.*

What the presenter appeared to be doing was reacting to the change in numbers in the room by comparison with the number of people already in the room. Had there been a hundred people at the beginning it is doubtful if any of the new arrivals would have been noticed. Although the arrival rate would have been the same, one every two minutes, the impact of the change would be far less as the constant change is proportionally less. Thus the impact of change can be defined as the ratio of the change to the initial base value, that is:

$$\text{impact of change} \triangleq \frac{\mathrm{d}x/\mathrm{d}t}{x}$$

where x is the variable being changed. This is a ratio measure of change, effectively the derivative of the logarithm of the variable $\mathrm{d}(\ln x)/\mathrm{d}t$.

Impact of Force on Motion

The previous definition of impact concerned the way a cause affects a change in the variable, impact on change, that is "velocity". Of more relevance to the loop impact method is the impact a force has on the "acceleration" of the variable. Thus the following definition is given for the impact of a force F on the changes in a variable x:

$$\mathrm{I}_{Fx} \triangleq \frac{\mathrm{d}^2x/\mathrm{d}t^2}{\mathrm{d}x/\mathrm{d}t} \tag{1}$$

that is, the ratio of the acceleration to the velocity of x, the derivative of the logarithm of the velocity[2]. Consider the following illustration:

A ball is thrown upwards into the air with initial velocity u, subject to constant gravitational force of acceleration $-g$. Air resistance is assumed negligible. Let x be the vertical displacement of the ball from the ground, then $\mathrm{d}^2x/\mathrm{d}t^2 = -g$. Thus $\mathrm{d}x/\mathrm{d}t = u - gt$, where t is time, and $x = ut - \frac{1}{2}gt^2$, assuming the ball starts at $x = 0$. Thus, from (1), the impact of gravity on the ball's motion is given by:

$$\mathrm{I}_{F_g\,x} = \frac{g}{gt - u}$$

Although gravity is constant, its impact on the motion of the ball is not constant. The impact is greatest as the ball is slowing down, near the top of its motion, because gravity is inducing greater curvature in the graph of x against time, figure 1. The impact is infinite while the ball is temporarily at rest, $t = u/g$, and then changes polarity as the ball starts falling. The sign of the impact indicates whether the force is reinforcing the motion, positive, or resisting it, negative.

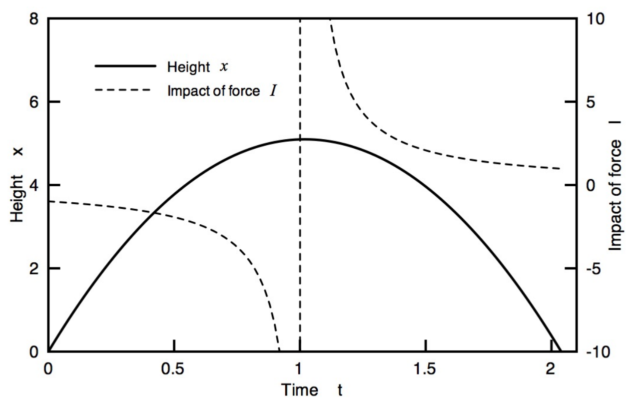

Fig. 1: Impact of force of gravity on vertical motion of a ball.

Physically the impact of a force is measuring the extent to which the force can change the motion of the object (indicated by its position variable), given that it is already in motion. For the thrown ball, when it is moving very fast, the constant gravitational force is inducing only a small change in the motion of the ball, whereas when the ball is moving slowly the change in its motion due to gravity is much larger, in percentage terms.

For example, if the ball is moving at 50 m/s, then the change of velocity in a tenth of a second is 2%. For a velocity of 10 m/s, the change is 10%, and for 2 m/s the change is 50%. Thus a constant force has more impact on a slow moving object compared with a fast one, as the change in the position of the object is more noticeable.

The impact of a force on motion can be viewed as a measure of curvature, similar to the radius of curvature used in analytical geometry. However, whereas the latter is constant when the curve is circular, the impact is constant when the curve is exponential. Consider a force that induces exponential acceleration, $\mathrm{d}^2x/\mathrm{d}t^2 = e^{at}$, for a constant a. Then $\mathrm{d}x/\mathrm{d}t = e^{at}/a$, and the impact of the force on x is $\mathrm{I}_{F_{\exp}\,x} = a$, a constant. The impact is positive if $a > 0$ reinforcing the motion, and negative if $a < 0$, resisting the motion. Impact is a more natural method of measuring curvature for dynamical motion, where linear processes generate exponential behavior.

Thus the impact of a force, in the sense defined here, is the impact on the *motion of an object*, units $[T^{-1}]$. It should not be confused with the impact an object makes when it collides with

[2]Care is needed with the mathematical sign in the derivative of the logarithm of a negative velocity.

another object, which is a force applied over a very short period of time, units $[MLT^{-2}]$, also called a shock.

Impact of a Stock on a Stock

Consider a stock y influencing a stock x, figure 2, equation (2):

$$\mathrm{d}x/\mathrm{d}t = f(y) \tag{2}$$

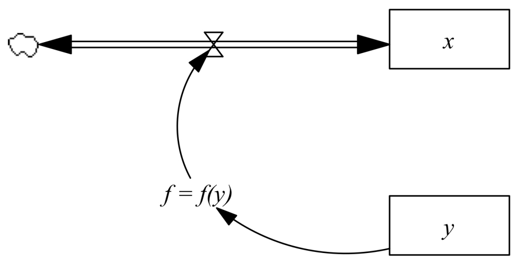

Fig. 2: One stock, x, influenced by another, y.

Following Hayward & Boswell (2014), and (1), then the impact of y on x is defined by:

$$\mathrm{I}_{\underline{yx}} \triangleq \frac{\mathrm{d}^2x/\mathrm{d}t^2}{\mathrm{d}x/\mathrm{d}t} = f'(y)\frac{\dot{y}}{\dot{x}} \tag{3}$$

where the underline subscript indicates the causal pathway of the impact. This definition is referred to as the pathway participation between y and x by Mojtahedzadeh et.al. (2004), where $f'(y)$ is the link gain (Kampmann, 2012). Although (3) is called loop impact in Hayward & Boswell (2014), it can be seen that the concept of impact is independent of the whether the link between the stocks is part of a feedback loop. Thus (3) will be called **Stock Impact** as it represents the effect on the acceleration of x due to changes in y.

Of course if a system has feedback loops then the stock impact will represent the impact of the loop on the stocks. Consider the general linear system with two stocks, figure 3, equations (4–5):

$$\dot{x} = ax + by \tag{4}$$

$$\dot{y} = cx + dy \tag{5}$$

with constants a, b, c, d. There are two first order feedback loops, L_1, L_2 with gains $G_1 = a$ and $G_2 = d$; and one second order loop, L_3 with gain $G_3 = bc$. Using definition (3), the stock impacts are given by:

$$\mathrm{I}_{\underline{xx}}(L_1) = a \tag{6}$$

$$\mathrm{I}_{\underline{yy}}(L_2) = d \tag{7}$$

$$\mathrm{I}_{\underline{yx}}(L_3) = \frac{b(cx+dy)}{ax+by} \tag{8}$$

$$\mathrm{I}_{\underline{xy}}(L_3) = \frac{c(ax+by)}{cx+dy} \tag{9}$$

where the loop identifier has been added to the notation for impacts in order to indicate the loop in which the pathway is embedded[3]. The stock impacts $\mathrm{I}_{\underline{xx}}(L_1)$ and $\mathrm{I}_{\underline{yy}}(L_2)$ are the direct impact

[3]Further examples of computing impacts analytically are found in Hayward & Boswell (2014) and Hayward (2012).

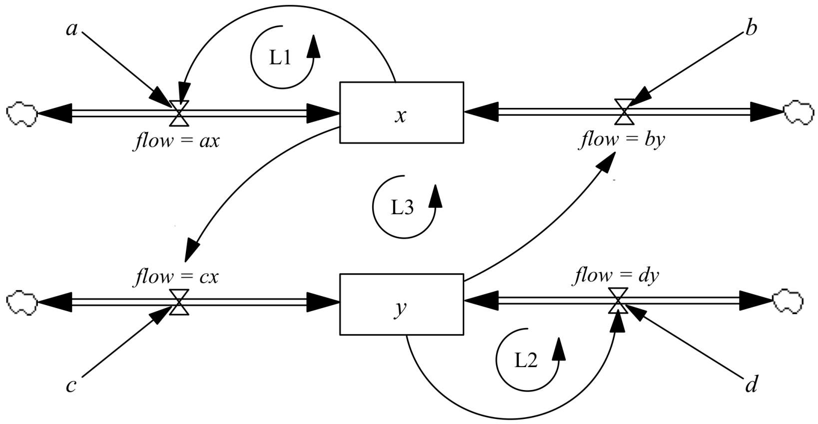

Fig. 3: Generic linear 2 stock system.

of a stock on itself, via first order feedback loops, L_1, L_2, and are equal to the loop gains. Thus it is natural to refer to them as <u>loop impacts</u>. Because the stock impacts $\mathrm{I}_{\underline{xy}}(L_3)$ and $\mathrm{I}_{\underline{yx}}(L_3)$ are part of a loop, L_3, they could also be called loop impacts, provided it is clear that they are the impacts of the loop on each stock, which in general will not be equal. Although neither are equal to the loop gain, their product does equal the gain, $\mathrm{I}_{\underline{xy}}(L_3)\mathrm{I}_{\underline{yx}}(L_3) = G_3 = bc$; a special case of the loop impact product result (Hayward & Boswell, 2014, appendix C), which shows that gain is the product of impacts in the loop regardless of model complexity and non-linearity. Note the polarity of the impacts (8–9) may flip sign, but do so together such that the polarity of the loop gain $G_3 = bc$ is preserved.

However the concept of impact is more general than that of feedback loops. Let $c = 0$ in (4–5) figure 3, which breaks the loop L_3 with $\mathrm{I}_{\underline{xy}} = 0$, where the loop indentify, $L3$ must now be dropped. y is now an exogenous influence on x, with impact $\mathrm{I}_{\underline{yx}} = bdy/(ax + by)$. This can no longer be referred to as loop impact as there is no loop, thus the term stock impact is preferred in this case. This impact can also flip polarity, although this time there is no loop gain to preserve.

With the concept of stock impact established, and with its definition, (3), the same as that of force impact, (1), the question is now asked as to whether the influence of one stock on another, as in figure 2, constitutes a force in the Newtonian sense. Differentiating (2) to obtain the acceleration of x gives: $\mathrm{d}^2x/\mathrm{d}t^2 = f'(y)\dot{y}$. Thus it is clear that y does not quantify a "force" affecting x. Rather it is the time derivative of the right hand side that is analogous to the Newtonian concept of force. Nevertheless stock impact is playing the same role as the impact of a force introduced earlier. Thus it can be stated that if a stock y affects a stock x (through the latter's flows), then y exerts a force on x, whose *impact* is measured the same way as a force, even though the force is not quantified by the value of y. These concepts will be used to draw out the Newtonian analogy in system dynamics in the next section.

It is noted that one significant difference between system dynamics, and Newtonian mechanics, is that in system dynamics stock variables can be in completely different units. In mechanics the state variables are position coordinates. always in the same units. By contrast in the second order model of figure 3, equations (4–5), there is no reason why x and y should be in the same units, which makes it difficult to compare the effects of forces on x with forces on y. However the impact of a force always has units of "per unit time", and thus are independent of the units of the stock imparting the force and the stock effected by the force. Thus forces from variables with any units can be compared through their impacts.

Before progressing it should be noted that the concept of measuring the impact of a variable by comparing the ratio of its second and first derivatives is not new, or confined to system dynamics. In functional data analysis this ratio is used as a growth factor in the analysis of time series data, where it is called the relative acceleration (Ramsay & Silverman, 2002). The ratio also occurs in the concept of financial risk aversion, where it is measures the curvature of a utility in an way that is invariant under certain transformations.(Pratt, 1964).

3 "Newton's" Laws of System Dynamics

In order investigate a Newtonian interpretation of system dynamics, some conventions will be assumed. Firstly, if a variable is stated as a stock then it is assumed to have flows, even if those flows do not explicitly appear in a model diagram. Although it is possible to include a stock with no flows in system dynamics software, in a finished model such a stock is constant, effectively a parameter/converter. The use of a stock will always indicate a variable which can be potentially changed through its flows.

Secondly, a flow with no connecting element is assumed to be constant in time, explicitly and implicitly, even though some software packages allow it to be a variable. If the intention is for a flow to vary over time then a connecting element should be used to indicate this. Any such variable element connected to a flow is deemed to originate in a force.

In what follows the stocks labeled by x, y etc. represent any system dynamics stock, such as population number, debt, inventory, motivation, burnout, etc. The three laws which Newton developed for mechanics are now widened to apply to any type of stock variable, and could be named: The three laws of stock dynamics. The section is developed abstractly with no particular application in mind.

Law 1 – Uniform Motion

A stock will remain level or change uniformly unless acted upon by a force.

This is the equivalent of Newton's first law of motion, applied to a single stock x, and represents the system in figure 4, with equation $\dot{x} = k$, where k is the net flow. The law applies regardless of the number of flows. The stock either stays at "rest" or in motion at the same "speed", unaffected by any force to either slow it down or speed it up. The graph of x against time will be linear, for example figure 5; the lack of curvature indicating no force. At this stage no concept of mass or momentum is required.

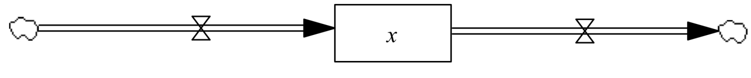

Fig. 4: Model of stock with no applied forces.

For the model in figure 4 to give unique behavior, both its initial value, x_0, and its initial net flow, that is its initial "velocity" $\dot{x}_0$, are required, as in Newtonian mechanics.

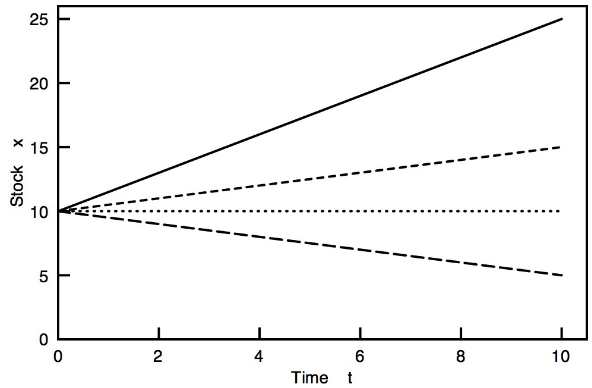

Fig. 5: Behavior of stock with no applied forces and four different net flows.

Law 2 – Change of Motion Due to Force

The acceleration of a stock produced by a net force is in proportion to that force and the inverse "mass" of the stock.

This is similar to one of the ways of Newton's second law is expressed, where the acceleration is proportional to the net force, and inversely proportional to the mass (Physics Classroom, 2015). In the system dynamics case the equivalent of the mass is inverse of the sensitivity of the stock to the force, the converter $a \equiv 1/m$, in figure 6. In the system dynamics model, the influence of the force on the stock appears as a converter, y, giving $\dot{x} = ay$. The measure of the force is $\mathrm{d}y/\mathrm{d}t$ as the equation for the acceleration is $\ddot{x} = a\dot{y} = \dot{y}/m$, with the impact of the force on the stock x the same as the impact of y, as noted in section 2. This equation is the equivalent of the mathematical expression of Newton's law of motion:

$$F = \frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{1}{a}\right)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \tag{10}$$

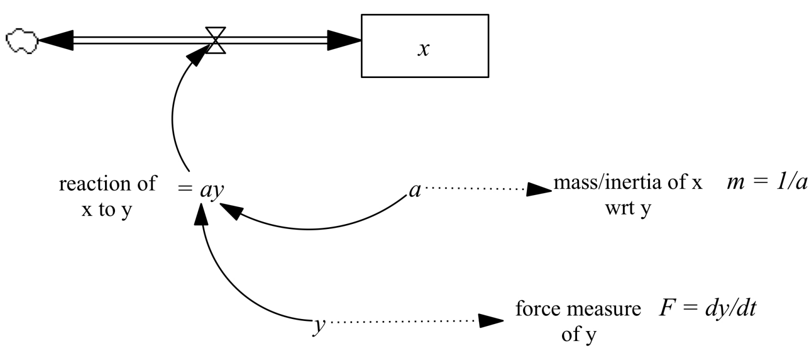

Fig. 6: Stock with applied force.

Consider the effect of a step change in y on x, at $t = 2$, for three different values of sensitivity a. The lower values of a means the force has less effect on the stock x, figure 7 (a), which can be interpreted as a stock with greater inertial resistance to the force. That is, a force has less effect on a higher mass stock. The measure of the force, $\dot{y}$, is a pulse, figure 7 (b), as is its impact $\ddot{x}/\dot{x}$. Thus a converter with a step change represents an impulsive force[4].

[4]The force is a delta function but appears as a finite spike in figure 7 (b) due to the fixed step length used in the numerical integration.

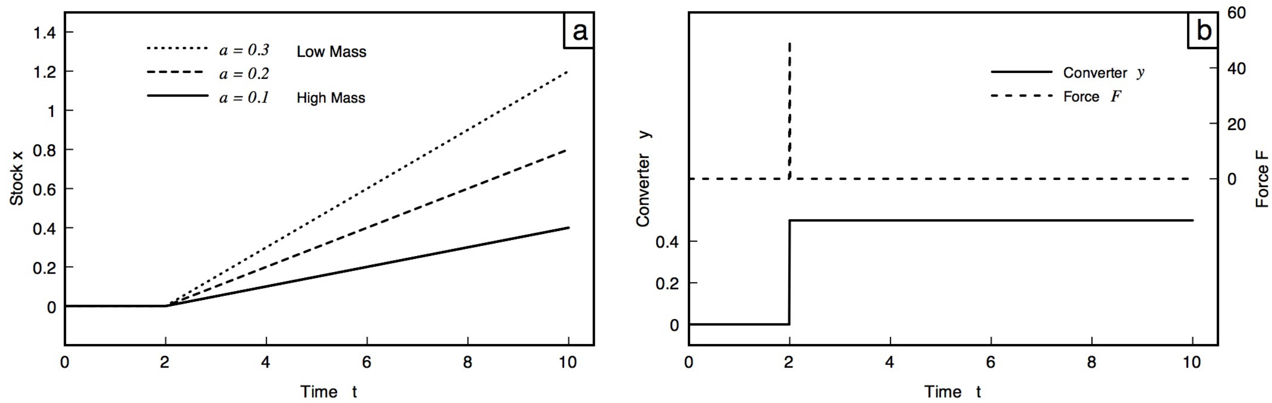

Fig. 7: (a) Effect of pulse force on stock x showing the inertial effect of high mass, low sensitivity a. (b) Step value in converter y is equivalent to pulse force $y = \mathrm{d}y/\mathrm{d}t$

It should be noted that the converter y is playing the part of momentum. The three runs for x in figure 7 (a) represent a system with the same momentum change with differing masses. Thus an alternative version of the second law can be given, that is the rate of change of stock momentum is proportional to the applied force. This interpretation will become more helpful when the applied force is due to an external stock/flow system.

Forces Due to Stocks

The most common way the concept of force will be used in a system dynamics model is when changes in a stock affect other stock/flow systems. For example, let a stock $y \geq 0$ exert a constant force on two other stocks x_1, x_2 with different sensitivities to y, that is different masses with respect to y, figure 8. The equations are $\dot{x}_i = a_i y$ for $i = 1, 2$ and $\dot{y} = k$, where k is a constant. Let $a_1 = 0.2$ and $a_2 = 0.4$, making x_1 the least sensitive of the two stocks to the common force. For a constant force $k = -4$, and $y_0 = 20$, both stocks are brought to rest by $t = 5$, with x_2 reaching the higher value, figure 9. The common force due to y has had more effect on x_2 due to it having less inertia than x_1 thus less resistance to change. For x_2 to have reached 10, like x_1, before stopping, a force of twice the magnitude would have been required.

The impact of the force due to y on each of the stocks $x1, x2$ is the same, figure 9, as it is a ratio measure, $\ddot{x}_i/\dot{x}_i = \dot{y}/y = -k/(y_0 - kt)$ for $t < 5$. The impact of y is independent of the sensitivities of x_1, x_2 and increases in magnitude as the force has more effect on the motion of the stocks. For $t > 5$ the impact is zero as both stocks are at rest, due to y having reached zero. y plays the role of the momentum of the stocks x_i.

Thus the interpretation of the the second law of motion in system dynamics is that a stock S that affects other stocks in the system represents the common momentum of the other stocks, with its flow being their common force. The differing effects S has on the other stocks is due to those stocks having a different inertial response, their "masses", the inverse of the rate multipliers; the "lighter" stocks have more response. However the impact of S, as defined by Hayward & Boswell (2014) is the same on each stock, as for each stock its acceleration is measured next to its own rate of change. Thus the impact of one stock on another can be seen as a scale-free measure of the force of one stock on another. This interpretation is independent of the stock connections being part of a loop, thus independent of the concept of feedback.

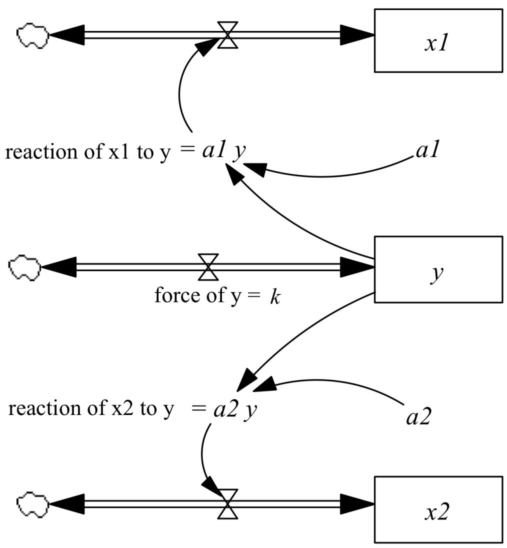

Fig. 8: Stocks with a common applied force.

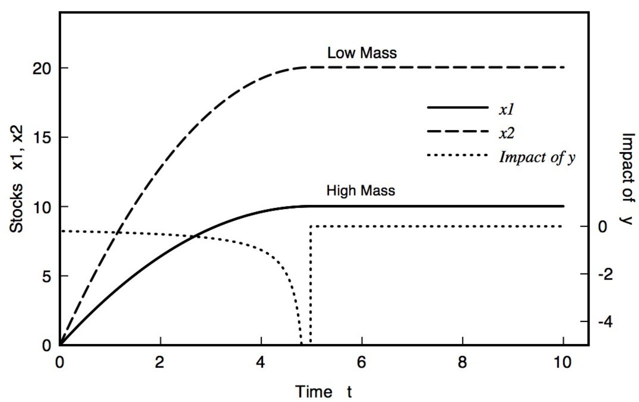

Fig. 9: Effect of stock y, representing a constant force, on other stocks x_1, x_2

Forces Due to Feedback Loops

When feedback is involved then a stock in a loop will be affected by a force, and be a force itself. Consider the second order linear system (4–5), figure 3. In the second order loop L_3, stock x exerts a force on y and y exerts a force back on x. Set $a = d = 0$ so that L_3 is the only loop, with gain $G_3 = bc$. Following the Newtonian analogy, the parameters b, c are the inverse of the masses of each stock with respect to the other. The gain is therefore the inverse of the product of the masses and thus represents the inertial resistance of the loop. If $G_3 < 0$ the system oscillates indefinitely with angular frequency equal to the square root of the $|G_3|$. Thus a higher "mass" loop will be more sluggish and oscillate slower.

In general the linear model has two first order loops, one on each stock. Set $b = 0$ in (4) to decouple the x stock from y, $\dot{x} = ax$. If $a < 0$ the loop is balancing, a draining process, which can also be seen as a form of frictional resistance $\ddot{x} = a\dot{x}$. Thus a first order loop can be interpreted as the force of a stock on itself. If $a > 0$ the loop is reinforcing, a compounding process, whose nearest mechanical analogy is the snowball gathering material and thus accelerating itself.

Thus each stock in the linear system is subject to two forces, whose balance will determine its behavior, as measured by their impacts. Because all the forces are part of loops these impacts can be called loop impacts on the understanding they are scale-free measures of the forces on each stock in those loops. The condition for stability depends on the gains: $G_1 + G_2 < 0$ and $G_1G_2 > G_3$ (see appendix). Thus for stability at least one the first order loops must be balancing.

Consider a system where L_2 is reinforcing and the other two loops balancing, such that the system is stable. The second order loop balancing, L_3, could be seen as an attempt to stabilize the system, for a given set of parameters. The loop impacts on each stock (6–9) are computed in the model and their dominance determined, figure 10(a).

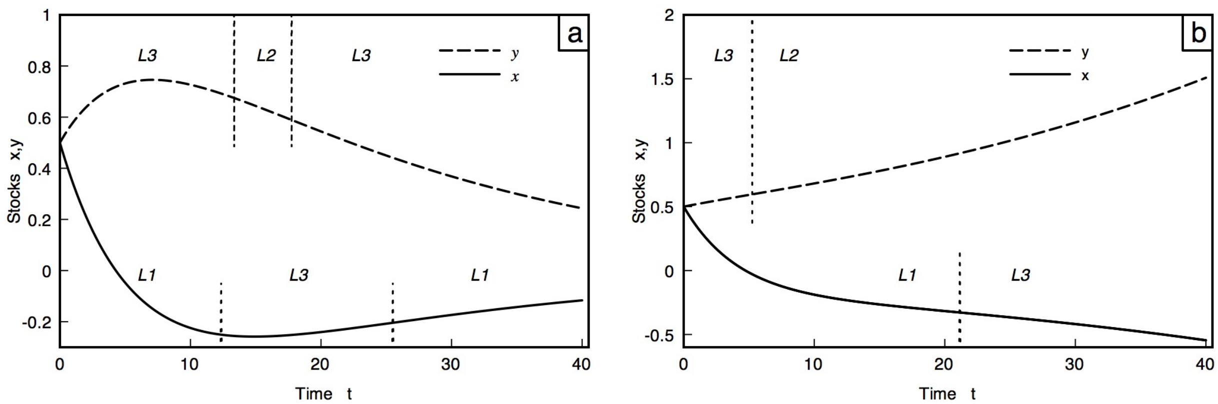

Fig. 10: Second Order Linear System (4–5) with $a = -0.25, b = -0.1, d = 0.03$, showing the change of force/loop dominance on each stock for the same loop structure: (a) Stable, with $c = 0.15$; (b) unstable, with $c = 0.01$.

Stock x starts with the "frictional" force, due to L_1 dominating, transitioning to L_3 at $t = 12.5$ and back to L_1 at $t = 25.5$, figure 10(a). The frictional force is a factor in bringing x to equilibrium, whereas in the middle period the impact of L_3 is destabilizing as it has changed polarity to positive (reinforcing) effect on x, figure 10(a), causing x to start accelerating until its impact is less than the magnitude L_1 again[5].

Initially stock y has L_3 dominating, and with a change of polarity changes the direction of y and causes it to accelerate downwards, $t = 7.2$. There is brief period where the reinforcing loop L_2 dominates, causing y to accelerate more, from $t = 13.4$. However L_3 ultimately dominates, from $t = 17.7$, and because it has changed polarity again to have negative (balancing) effect on y, and y is brought to stability.

That the loop L_3 is balancing and of constant gain is not a good indication of stock behavior as the impact of its force on each of the stocks is variable and changes polarity. There is a brief period where L_3 is dominant on both stocks, $t = 17.7 - 25.5$, and could be said to dominate the system; but it is the balance of forces on each stock that is key to their behavior. Stability comes from the final dominance being a force with negative impact. Thus stability has been achieved through sufficient friction being applied to x, which is slowed enough to control y.

The system can be destabilized by reducing c to 0.01. The loop structure remains identical, but now $G_1G_2 < G_3$ and the system is unstable with $x \to -\infty$, $y \to \infty$. For y, the force due to x, via L_3 is unable to regain dominance over that of L_2, the destabilizing force. For x, the

[5] A second order balancing loop will always have one of its impacts positive, with the other negative to maintain a negative gain. Thus one stock in the loop is always being accelerated and must have another balancing force of greater magnitude if stability is to be achieved.

loop L_1 is unable to regain dominance over L_3, as is there is insufficient friction to counteract the force from y. The reduction in the absolute value of the gain G_3 can be interpreted as the loop L_3, and thus the system, having more inertia, that is more mass, and thus less effective in controlling the destabilizing influence of L_2.

If instead the the gain of L_3 were increased, the system remains stable, but will oscillate first as $(G_1 - G_2)^2 < -4G_3$ (see appendix). This could be interpreted as a less massive system being over responsive to the corrective effects of L_3 as the frictional dissipation L_1 is relatively less effective.

Thus the loop impact method of Hayward & Boswell (2014) can also be understood in terms of force dominance on each stock, with the use of Newtonian terms such as mass, inertia and friction providing an alternative explanation of behavior.

Law 3 – Equal and Opposite Forces

The force on a stock through a flow has an equal and opposite force on a stock at the other end of the flow.

For example, consider a stock x with a draining process flowing into stock y, figure 11 with equations $\dot{x} = -ax$, $\dot{y} = ax$. Not only is the flow conserved, but the forces are equal and opposite in effect, $\ddot{x} = -\ddot{y}$. Their impacts are identical $\mathrm{I}_{\underline{x}x}(B) = \mathrm{I}_{x\underline{y}}(B) = -a$, as the sign of the force is also determined by the sign of the flow on each stock. As such the behavior of x and y mirror each other, both decelerating to stability, assuming $x, y \geq 0$, figure 12. Examples of this mirroring effect of Newton's third law can be seen in the epidemic SIR model (Hayward, 2012).

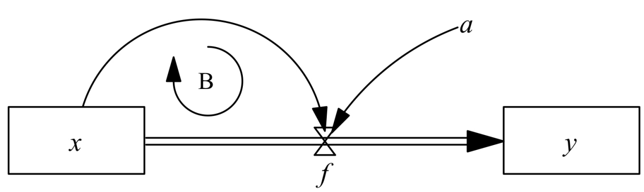

Fig. 11: The force due to loop B has an equal and opposite effect on each stock.

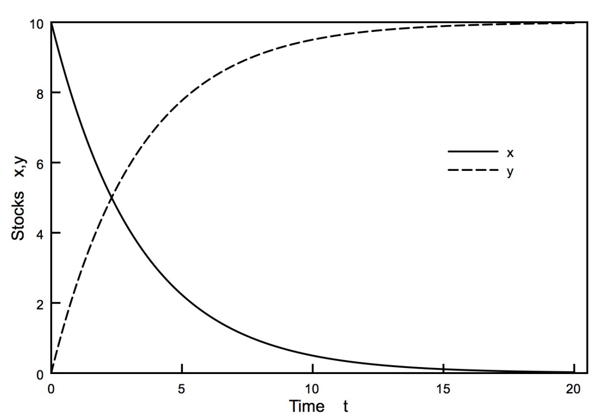

Fig. 12: Equal and opposite forces on stocks in a conserved flow, with same impact on behavior.

4 Application to Inventory-Workforce Model

To help understand how Newtonian concepts can be applied to a system dynamics model, consider a standard two state inventory-workforce model subject to an exogenous demand on sales, figure 13 (Ventana, 2011). The equations are given on the system dynamics diagram. There are two forces on the inventory, one from demand, and one directly from workforce connected with loop $B2$. There are three forces on the workforce: the frictional self force connected with $B1$; one from the inventory via the inventory adjustment control, connected with $B2$; and one from demand via sales which is not connected with a feedback loop; an external force in the Newtonian sense.

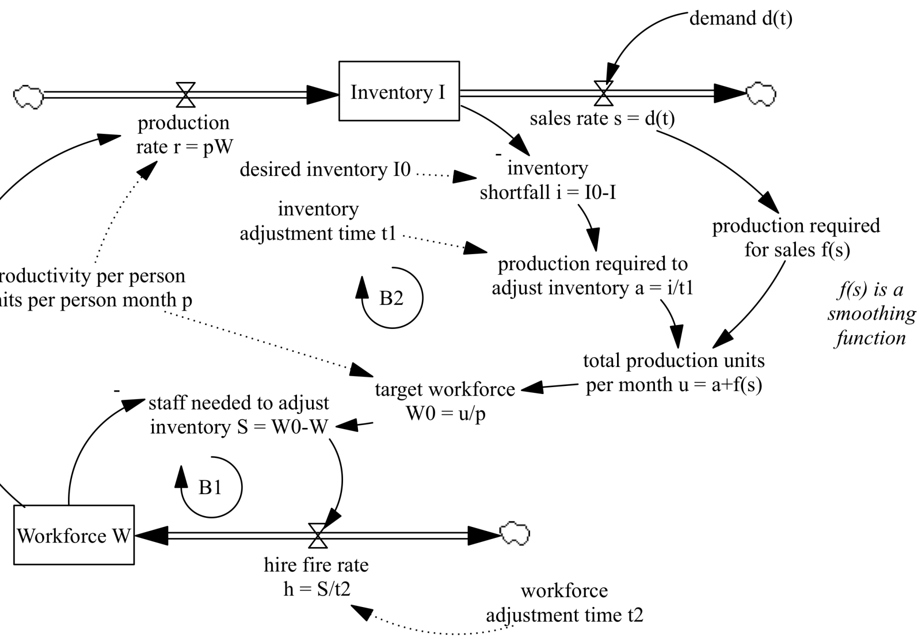

Fig. 13: Two state inventory-workforce model. (Ventana, 2011)

Let the system start in equilibrium with the inventory at 100, the workforce at 20, and a demand of 10. Let demand rise steadily from 10 to 12 between $t = 5$ and $t = 10$. The force exerted by the demand only applies during this period, and is the slope of the line, figure 14(a). Thus demand has an impact on the inventory from $t = 5$, however at $t = 9.1$ the restoring force from the workforce, part of loop $B2$, exceeds the demand impact and dominates the rest of the inventory's return to equilibrium.

The more involved dynamics takes place on the workforce. The impacts of the three forces on the workforce can be computed numerically, using the method of Hayward & Boswell (2014), and the transitions indicated on the graph, figure 14(a). Initially, from $t = 5$, $B2$ is dominant as the workforce increases to match the production required by the inventory. However there is also a small but increasing effect from the exogenous demand. By $t = 9.2$ it takes both $B2$ and the demand forces, indicated by D, to dominate the behavior of *workforce*, thus showing the corrective action from sales is helping to adjust the the desired workforce. This continues until $t = 10.9$ when the frictional force of the workforce, due to the stock adjustment loop $B1$ starts to slow its growth. That the demand continues to have an impact on the workforce after it has stop changing, $t = 10$, is due to the delays in *production required for sales*.

At $t = 13.5$ the corrective force $B2$ again dominates causing the workforce to peak, and accelerate

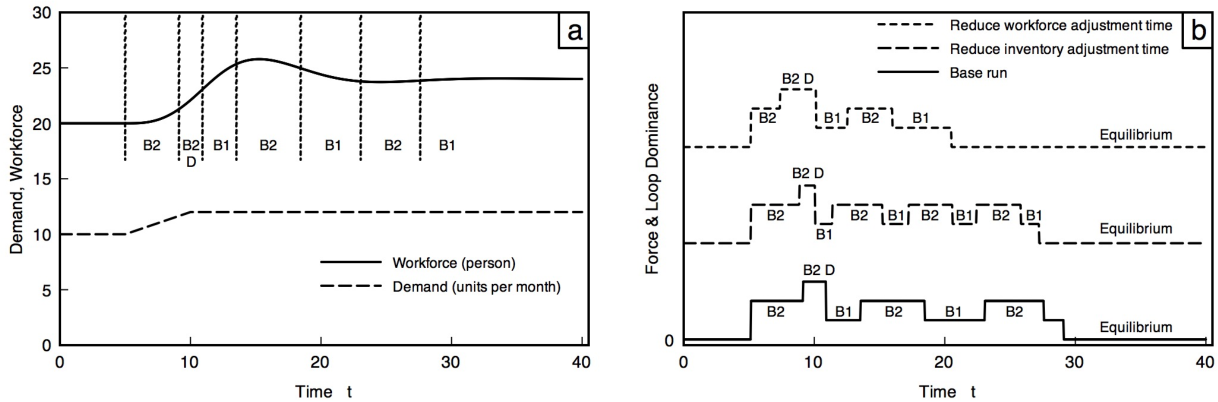

Fig. 14: Results of inventory-workforce model. D indicates dominance of exogeneous demand $d(t)$. (a) Base run for *workforce* with transitions of force/loop impacts, $t_1 = t_2 = 2.5$, $p = 0.5$, $I_0 = 100$. (b) Force/loop dominance on *workforce*, base run compared with reducing inventory and workforce adjustment times, $t_1 = 1$ and $t_2 = 1$ respectively.

downwards. The impacts of B2 on the two stocks has changed polarity with the change in direction. The remainder of the motion is repeated change between $B2$ and $B1$. The latter is a dissipative force damping the oscillations caused by $B2$.

Let the scenario in figure 14(a) be the base run, and consider the differing effects of reducing the inventory adjustment time t_1 and workforce adjustment time t_2. Using the Newtonian analogy, t_2 controls the friction, and t_1 effects the inertia of the second order loop B_2. Figure 14(b), compares the effects of adjustment time reduction on the force/loop impacts on the stock *Workforce* using the loop picker algorithm of Hayward & Boswell (2014).

Reducing inventory adjustment time brings in the first occurrence of $B1$ dominance earlier, compared with the base run. However its appearance is too brief to give sufficient correction as the impact of $B2$ has increased. This has happened because the inventory adjustment time t_1 contributes to the "mass" associated with the loop $B2$. It now has less inertia and is thus harder to control. The resulting effect is a greater number of oscillations compared with the base run, figure 14(b) middle run, even though it reaches equilibrium slightly faster[6].

Reducing the workforce adjustment time, figure 14(b) top run, also has the first occurrence of $B1$ starting earlier than the base run. However this has been achieved by reducing the duration of $B2$ dominance, as $B1$ is now stronger compared with $B2$. The weakness of $B2$ can be seen in the first period of its dominance, which needs a longer period of assistance from the demand, $B2\,D$, to overcome $B1$. This results in fewer oscillations and thus equilibrium is reached faster. In Newtonian terms this can be interpreted as increasing friction at the expense of the corrective and exogenous forces.

5 Notational Refinement

Further insight into the Newtonian understanding of the previous model can be obtained by computing the impacts of the three forces analytically, following the method of Hayward & Boswell (2014). This method requires the system dynamics model to be reduced to differential

[6] A threshold is required to stop registering forces when they get very small. This defines the numerical approximation to equilibrium.

equation form and the definition of stock impact (3), to be applied to each stock on the right hand side of the each equation according to causal pathway. Unfortunately the differential equation form of the system dynamics model does not retain the pathway information.

For example the differential equations for the inventory-workforce model, figure 13, are:

$$\dot{I} = pW - d(t) \tag{11}$$

$$\dot{W} = \frac{1}{t_2}\left[\frac{1}{p}\left(\frac{I_0 - I}{t_1} + f(d(t))\right) - W\right] \tag{12}$$

There is no indication in this model that the W in equation (13) proceeds via a different pathway to the W in equation (14). (Likewise the pathway from the exogenous $d(t)$.) In this model the issue is resolved by the Ws being in different equations, i.e. the different target stocks, but in models where there are multiple pathways between the same stocks there will be no such clear distinction.

Thus a notation is proposed so that the differential equations are written in a form as to retain the causal network information. Using the inventory-workforce model as an example consider the causals link from *Inventory* stock I to the auxiliary variable *production required to adjust inventory* a, via i, the *inventory shortfall*, figure 13. The equation can enhanced with the subscript of any intermediary variables, As $i = I_0 - I$ and $a = i/t_1$, then substituting, $a = (I_0 - I_{\underline{i}})/t_1$, where the underline on the subscript indicates it is used for a causal pathway. Thus the formula shows I connects to a via i.

The next auxiliary variable in the pathway is *total production units per month*, $u = a + f(s)$. Using the pathway subscript notation this is written $u = (I_0 - I_{\underline{ia}})/t_1 + f(s)$, where both intermediary variables have been subscripted on the causal stock I. Thus the formula now shows I connects to u via i and a. The general form of this process is given in appendix B.

Continuing this process to the flow h of the target stock W, will give

$$h = \frac{1}{t_2 p}\left(\frac{I_0 - I_{\underline{\underline{iauW_0S}}}}{t_1}\right)$$

with the underlined subscript on I preserving the entire causal pathway from cause I to target W through its flow h. Thus using the equations of figure 13 the differential equations in network form of the inventory-workforce model are:

$$\dot{I} = pW_{\underline{r}} - d_{\underline{s}}(t) \tag{13}$$

$$\dot{W} = \frac{1}{t_2}\left[\frac{1}{p}\left(\frac{I_0 - I_{\underline{\underline{iauW_0Sh}}}}{t_1} + f(d_{\underline{sfuW_0Sh}}(t))\right) - W_{\underline{Sh}}\right] \tag{14}$$

The subscripts of I, $d(t)$ and W in equation (14) clarify which sections of a causal chain they have in common, and where they differ. Thus these "networked" differential equations preserve the causal topology of the system dynamics model figure 13.

The impacts of the stocks, and exogenous demand, on a target stock, can be derived by differentiating along the appropriate pathway. For example, following (3), the impact of I on W though the inventory shortfall pathway is defined as the partial derivative connected with the W of that network.

$$\mathrm{I}_{\underline{IiauW_0ShW}} \triangleq \frac{\partial \dot{W}}{\partial I}\Bigg\|_{\underline{iauW_0Sh}} \times \frac{\dot{I}}{\dot{W}} \triangleq \frac{\partial \dot{W}}{\partial I_{\underline{iauW_0Sh}}}\frac{\dot{I}}{\dot{W}} \tag{15}$$

The double vertical line in (15) indicates the causal pathway derivative. A general definition is given in appendix B.

Thus the three impacts on stock W are:

$$\mathrm{I}_{\underline{IiauW_0ShW}}(B2) = \frac{pW - d(t)}{I_0 - I + t_1 f(d) - t_1 pW} \tag{16}$$

$$\mathrm{I}_{\underline{dsfuW_0ShW}} = \frac{t_1 f'(d)\dot{d}(t)}{I_0 - I + t_1 f(d) - t_1 pW} \tag{17}$$

$$\mathrm{I}_{\underline{WShW}}(B1)) = -\frac{1}{t_2} \tag{18}$$

Because the paths are not shared by loops the subscripts can be omitted without confusion, and loop identifiers added, $\mathrm{I}(B2), \mathrm{I}_{\underline{d}}, \mathrm{I}(B1)$ respectively[7].

Returning to the analysis of the inventory-workforce model, both $\mathrm{I}(B2)$ and $\mathrm{I}_{\underline{d}}$ are independent of the workforce adjustment time, t_2, thus it is possible to increase the frictional force without any direct effect on these impacts, just the indirect effect through W; hence the success of that policy on reducing oscillations. Reducing t_1 increases $\mathrm{I}(B2)$, the reduction in inertia in this loop referred to earlier. $\mathrm{I}_d/\mathrm{I}(B2) \propto t_1$ showing that reducing inventory adjustment time, weakens the effect of the target of sales compared with production. The resulting higher relative impact of $B2$ on W increases the number of oscillations.

6 Application: Economic Long-Wave Model

For a more challenging application of the Newtonian view of system dynamics consider Sterman's (1985) economic long-wave model. The equations and parameter values are taken from Kampmann (2012) and it is assumed the reader has some familiarity with the model, which has become one of the benchmarks for analytical methods. The analysis given here is just an initial exploration as to whether there are Newtonian aspects to the model's behavior, and as such, the investigation concentrates on the mathematical interpretation alone, rather than the conceptual meaning of the model.

The stock/flow diagram of Kampmann (2012) is given in figure 15, where his equations of table 2 are embedded in the diagram[8]. Although the model has only 3 stock variables, the connections are complex with 36 loops, of which 16 are independent (Kampmann, 2012). More than one independent loop set can be chosen because many of the loops share edges in parts of their structure.

[7] Even if the subscripts had been retained some notational simplification would be possible by combining adjacent auxiliaries where there is no branching or merging of pathways.

[8] A typographical error in the depreciation equation of Kampmann's (2012) paper is corrected using his accompanying Vensim model, $d = K/\tau$; denominator is τ rather than δ.

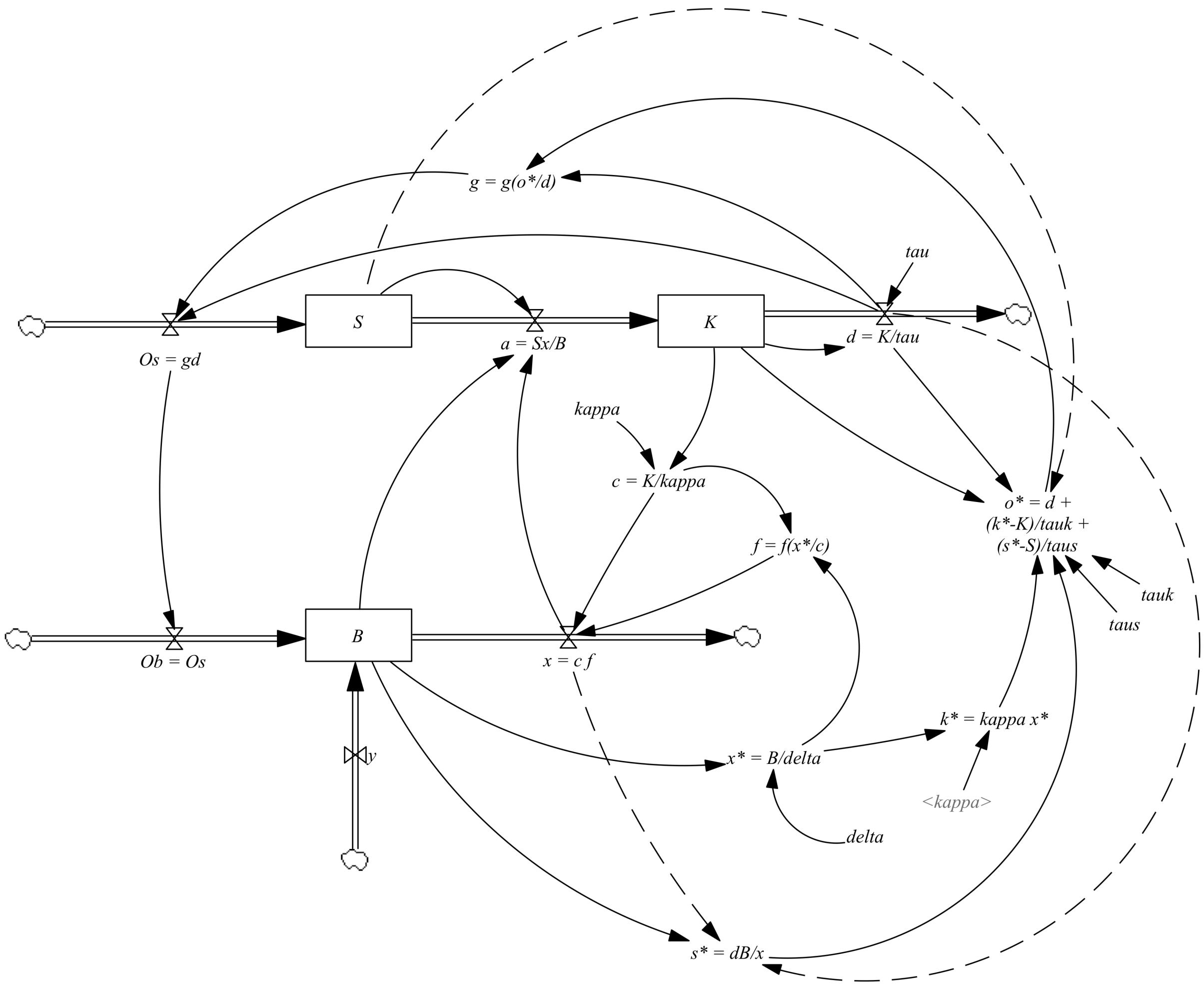

Fig. 15: Economic long-wave model, with model equations given on each stock, flow and auxiliary. f and g are graphical converters, expressed in functional form. Some connectors are dashed for readability.

Following the method of the previous section the model is written as three networked differential equations, using figure 15 and (Kampmann, 2012, table 2):

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{S_{\underline{a}}K_{\underline{cxa}}\, f\left[\frac{\kappa B_{\underline{\underline{x^*fxa}}}}{\delta K_{\underline{cfxa}}}\right]}{\kappa B_{\underline{a}}} - \frac{K_{\underline{d}}}{\tau} \tag{19}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{K_{\underline{do}}}{\tau} g\left[\frac{K_{\underline{do^*go}} + \frac{\kappa\tau B_{\underline{\underline{x^*k^*o^*go}}} - \delta\tau K_{\underline{\underline{o^*go}}}}{\delta\tau_k} + \frac{\frac{\kappa K_{\underline{ds^*o^*go}} B_{\underline{s^*o^*go}}}{K_{\underline{cxs^*o^*go}}\, f\left[\frac{\kappa B_{\underline{\underline{x^*fxs^*o^*go}}}}{\delta K_{\underline{cfxs^*o^*go}}}\right]} - \tau S_{\underline{o^*go}}}{\tau_s}}{K_{\underline{dgo}}}\right] - \frac{S_{\underline{a}}K_{\underline{cxa}}\, f\left[\frac{\kappa B_{\underline{\underline{x^*fxa}}}}{\delta K_{\underline{cfxa}}}\right]}{\kappa B_{\underline{a}}} \tag{20}$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{K_{\underline{\underline{do}}}}{\tau} g\left[\frac{K_{\underline{\underline{do^*go}}} + \frac{\kappa\tau B_{\underline{\underline{x^*k^*o^*go}}} - \delta\tau K_{\underline{\underline{o^*go}}}}{\delta\tau_k} + \frac{\frac{\kappa K_{\underline{ds^*o^*go}} B_{\underline{s^*o^*go}}}{K_{\underline{cxs^*o^*go}}\, f\left[\frac{\kappa B_{\underline{\underline{x^*fxs^*o^*go}}}}{\delta K_{\underline{cfxs^*o^*go}}}\right]} - \tau S_{\underline{o^*go}}}{\tau_s}}{K_{\underline{dgo}}}\right] + y - \frac{K_{\underline{\underline{cx}}}}{\kappa} f\left[\frac{\kappa B_{\underline{\underline{x^*fx}}}}{\delta K_{\underline{cfx}}}\right] \tag{21}$$

From a Newtonian point of view, the number of forces on each stock due to other stocks is unique. For example K has 6 forces, indicated by the 6 different pathway subscripted variables in (19). The use of subscripted variables prevents any algebraic reduction taking place through factoring or cancellation of stocks, unless they have come via the same pathway, Thus the networked differential equations (19–21) preserve the network topology of the system dynamics model, as explicitly expressed in the stock/flow diagram, and implicit in the model equations (Kampmann, 2012, table 2). Stocks are denoted by upper case letters to make them easier to identify. Note that the forces from $K_{\underline{ds^*o^*go}}$ and $K_{\underline{cxs^*o^*go}}$ identically oppose each other, as the K's would cancel in the fraction had they not been annotated by the path, (equations 20–21, last term of numerator in g's argument).

Thus equation (19) shows that K has 6 forces. 3 are self forces: via the flow d, ($K_{\underline{d}}$); via x directly, ($K_{\underline{cxa}}$); and via c through the graphical function f, ($K_{\underline{cfxa}}$). These are first order loops. There is one force from S via the flow a, which is the equal and opposite reaction (Newton's third law) from the draining loop on S. Finally, there are 2 forces from B, one directly via a and one through the function f.

The forces from S and B on K are all part of a number of higher order loops. Although there is some choice of the loop with which they can be identified, the forces of S and B on K are unique. Thus 6 unique stock impacts on K can be derived representing all the forces, and their relative strengths compared. The impacts can either be computed numerically in the simulation model, following the method of Hayward & Boswell (2014), or they can be derived analytically by differentiating the differential equations (19–21) by the appropriate variable, and ensuring the differentiation only occurs along the path of the force whose impact is being determined[9].

[9] In the subsequent analysis the numerical method was used with the analytical formulae providing a check.

For example, using the stock impact definition (3, 15), the impact of B on K via pathway $\underline{B_{x^*fxa}}$ is given by:

$$\mathrm{I}_{\underline{B_{x^*fxa}K}} = \left(\frac{\partial \dot{K}}{\partial B} \bigg\|_{\underline{x^*fxa}} \right) \frac{\dot{B}}{\dot{K}} = \left(\frac{\partial \dot{K}}{\partial B_{\underline{x^*fxa}}} \right) \frac{\dot{B}}{\dot{K}} = \frac{SKf'}{\kappa B} \frac{\kappa}{\delta K} \frac{\dot{B}}{\dot{K}} = \frac{Sf'}{\delta B} \frac{\dot{B}}{\dot{K}} \tag{22}$$

where all variables are held constant except for B along the $\underline{B_{x^*fxa}}$ pathway. Once the differentiation is performed the subscripts on the right hand side can be dropped and algebraic manipulation can take place. However the subscript on the impact symbol I, on the left hand side, is essential in order to precisely locate the path from the stock causing the force. The use of a loop identifier in this model is less helpful as some paths are shared between loops.

A simulation is performed and the variables examined from $t = 128$, once the limit cycle behavior is established. The change of dominance of stock impacts on K are given in figure 16(a). Growth is dominated by $\mathrm{I}_{\underline{SaK}}$, that is the reaction of K to the outflow of S. The only exception is a brief period from K itself via the $KcxaK$ pathway, enhancing the accelerated growth. Thus it can be said that the growth in K is largely a reaction to the outflow of S, that is a result of Newton's third law of motion. The same reaction force governs the change to decline of K and the acceleration that immediately follows. This cause of behavior is standard in chain models (Hayward & Boswell, 2014). The remainder of the decline is caused by the frictional dissipation force, impact $\mathrm{I}_{\underline{KdK}}$.

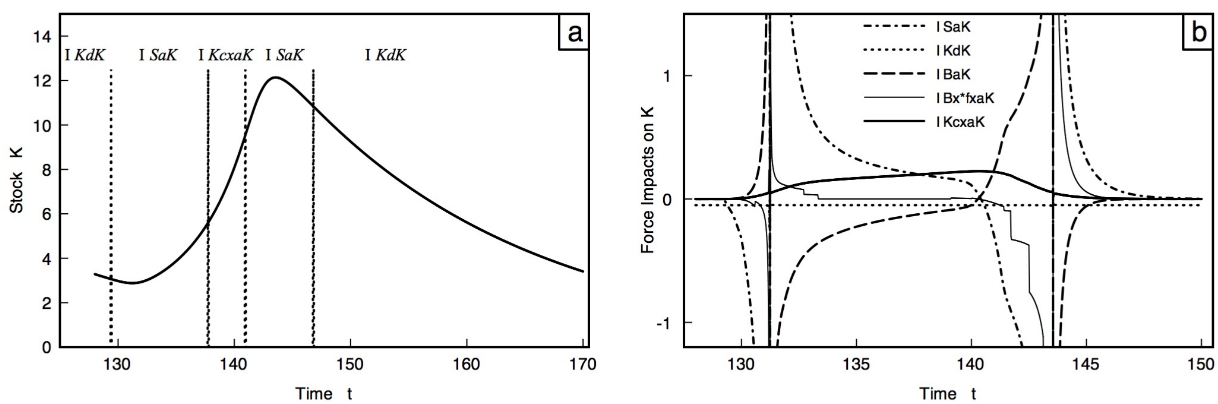

Fig. 16: Economic long-wave model with $\tau_k = \tau_s = \delta = 1.5$, $\tau = 20$, $\kappa = 3$ and $y = 1$. (a) Stock K with regions of force (stock impact) dominance. (b) Stock impacts for 5 of the 6 forces on K. $\mathrm{I}_{\underline{KcfxaK}}$ is too small to show; all bar $\mathrm{I}_{\underline{KaK}}$ are zero during K's decline.

Comparing the impacts of the forces, figure 16(b), shows the frictional force $\mathrm{I}_{\underline{KdK}} = -1/\tau$, is a small constant, but that all the other forces collapse to near zero throughout K's decline allowing $\mathrm{I}_{\underline{KdK}}$ to dominate. The two forces via the graphical function f are very small, except for $\mathrm{I}_{\underline{Bx^*fxaK}}$ near turning points only[10], even then it is swamped by the other forces. Other forces, such as $\mathrm{I}_{\underline{BaK}}$, are non-zero and do affect the motion, but they are not strong enough to explain the type of curvature, the acceleration and deceleration; they only influence its extent. Thus K's behavior can generally be explained by the relative effects of a reaction to a force on S (Newton's third law) and friction on K.

For the stock S there are 16 forces: 4 direct forces; 2 via f; and 10 via g. To simplify the understanding, the 2 forces via f are ignored as these pathways have already been shown to be

[10]The impact is not smooth due to f being based on a look-up table. Replacing f with a smoothed function would eliminate the sudden changes.

small, figure 16(b), and the ones via g are treated as a single combined force. Thus 5 impacts on S can be compared, figure 17(a). Essentially there are four phases: a short period of growth, due to a short impulse from g, about $t = 130$; a period of steady growth, where a number of forces dominate in combination (given by the arrow on the figure); a short impulse from g to cause S to decelerate, after $t = 140$; and its subsequent decline through $\mathrm{I}_{\underline{SaS}}$, the frictional force on S. The force dominance in the second period, the steady growth in S, is largely spurious as the forces through g during this period are zero, and the remaining four forces, $\mathrm{I}_{\underline{KdoS}}$, $\mathrm{I}_{\underline{SaS}}$, $\mathrm{I}_{\underline{KcxaS}}$, $\mathrm{I}_{\underline{BaS}}$, though not zero, nevertheless balance to almost zero, figure 17(b) (labelled *Sum of other forces* on the figure). Instead this second phase of steady growth is best explained by Newton's first law of motion, with S increasing under its own momentum after the impulse caused through g in the first phase. g only has effect in two short phases because all forces through it have their impacts proportional to $\dot{g}$, and most of the time g is either zero, or at saturation, thus horizontal with no gradient, figure 17(b).

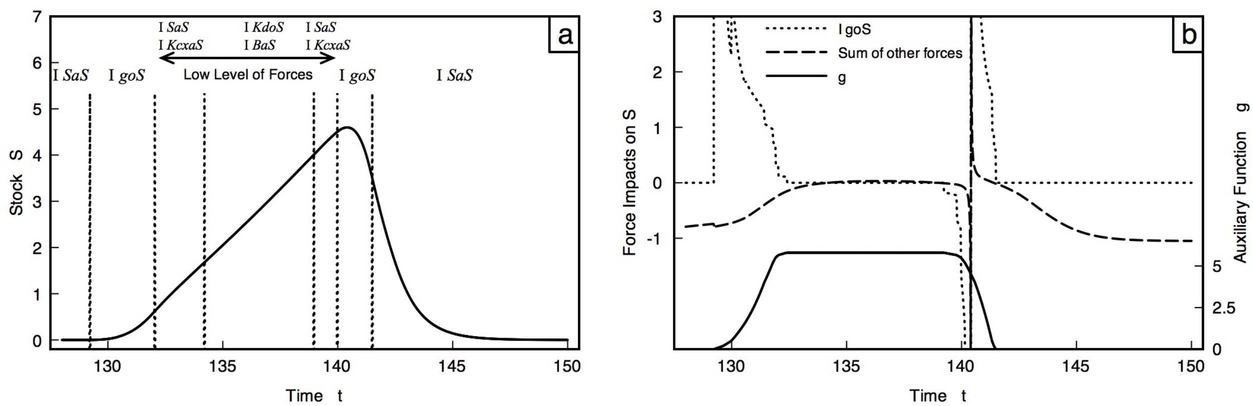

Fig. 17: Economic long-wave model. (a) Stock S with regions of force dominance, negligible force on region marked by arrow. (b) Force impacts for those through g, ($\mathrm{I}_{\underline{goS}}$), compared with the sum of the other 4 forces on S, ($\mathrm{I}_{\underline{KdoS}} + \mathrm{I}_{\underline{SaS}} + \mathrm{I}_{\underline{KcxaS}} + \mathrm{I}_{\underline{BaS}}$), showing a period of negligible force, $t = 132..139$. Graph g against time, showing it has no force (gradient) except at critical turning points of S.

Examining the impacts of the 10 forces through g shows that most are negligible. The rapid acceleration of S about $t = 130$ is dominated by $\mathrm{I}_{\underline{Bk^*o^*gos}}$, the target setting for K. The change from growth to decline of S about $t = 140$ is dominated by $\mathrm{I}_{\underline{Ko^*goS}}$ (initially assisted by $\mathrm{I}_{\underline{KcxaS}}$), with the following short period of deceleration again dominated by $\mathrm{I}_{\underline{Bk^*o^*gos}}$. Both $\mathrm{I}_{\underline{Bk^*o^*gos}}$ and $\mathrm{I}_{\underline{Ko^*goS}}$ are connected with the capital K adjustment process. Thus the dramatic changes in S are caused by two brief, but intense, periods of acceleration and deceleration caused by the capital adjustment process, with the remainder of the behavior either following Newton's first law, uniform growth, or frictional dissipation, giving exponential decline. That g is effectively acting like a switch, goes some way to explaining the severe non-linearity of the limit cycle.

The impacts on B, the backlog can be examined, though these are not included for lack of space. It is hoped that sufficient insight has been given to show the value of using Newtonian concepts, particularly that of force, to explain behavior from structure in a complex system dynamics model, with numerous loops.

7 Conclusion

The paper developed the concept of loop impact, proposed by Hayward & Boswell (2014), using the analogy with Newtonian mechanics. It was possible to identify the force exerted by one stock on another as the net rate of change of the stock representing the force. Key to the analogy is the concept of *impact* as a ratio measure of acceleration, which is the same for a mechanical force and the effect of one stock on another. These concepts correspond with the definition of loop impact, but are also applicable to exogenous influences. Newton's three laws of motion have their analogy in system dynamics and, together with the concept of mass, inertia and momentum, have a natural interpretation which can assist with understanding model behavior. The ideas were applied to an inventory-workforce model, and the economic long-wave model, where Newtonian interpretations were given to model behavior by examining the dominant forces on individual stocks. However it is not the value of the forces that determines dominance, but their impact as a ratio measure of the force compared with the net flow of the stock being changed. A notation was developed for the model differential equations that allowed the network topology of the system dynamics model to remain intact, enabling easier analytical computation of the impacts on stocks.

The word "force" has a precise meaning in physics which does not always transfer to social systems. Consider the following definition of "social force":

> *A social force is an element of society which has the capability of causing cultural change or influences people* (Business Directory, 2015).

The definition gives the impression that it takes a change in the force to give a change in the culture, that is if the force remains constant then the social variable in the culture affected by it remains fixed. If the social force were removed, the social condition would fall back to its original value prior to the force being acted, rather than continue under its own momentum. This is an almost Archimedean, or pre-Newtonian idea of force. The concept is about balance rather than acceleration. However, this can be replicated in system dynamics using the Newtonian understanding if both the stock representing the force, and the one it accelerates, have natural frictional dissipative forces, figure 18. In this case a change in the level of the force k will cause y to increase to a new limit, and thus x will rise, initially accelerating, and then slowing to reach a higher equilibrium. In this sense such a force could be said to cause a cultural change, whilst still imparting acceleration. Such natural dissipative forces have been propsed for psychological and soft variables by Levine (2000) and Hayward, Jeffs, et. al. (2014).

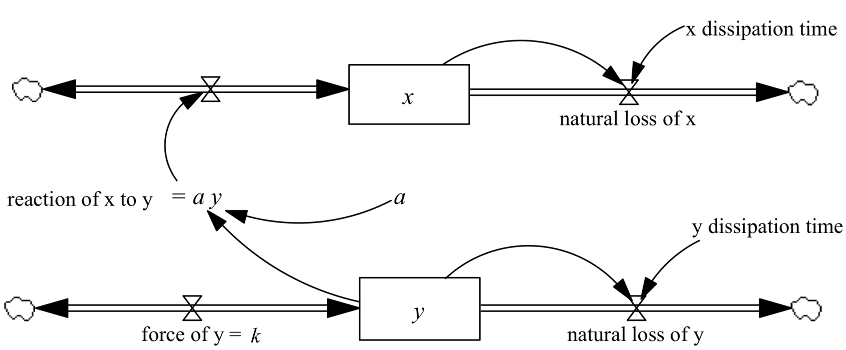

Fig. 18: Social force y acting on social variable x, including internal dissipative forces.

The work in this paper is an initial attempt to provide another approach to understanding the connection between system structure and behavior. It is seen as an enhancement to the feedback understanding of a model rather than a replacement. In many models the stock impacts are a measure of the effect of feedback on a stock, and give an indication of loop dominance where loops can be easily be identified, as in the inventory-workforce model. In a model as complex as the economic long-wave, there may be a number of loops that can be associated with a given stock impact. Thus, provided a suitable independent loop set can found, the impacts can be applied to the force effects of feedback loops on stocks.

The proposed method, and its Newtonian interpretation, needs to be applied to a wide range of models in order to investigate the extent to which the analogy can help understand model behavior. It is hoped that the work presented in this paper gives encouragement for further research into a Newtonian understanding of model behavior and its connection with system structure. An additional benefit of a Newtonian approach to system dynamics, and the use of the mathematical formalism presented, is that it may encourage researchers in mathematical modeling, and social physics, to consider system dynamics as a serious analytical approach to modeling that would enhance traditional non-linear analysis.

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Appendix A: Second Order Linear System

For a second order linear system the criteria for stability of the equilibrium can be expressed in terms of loop gains. The Jacobian J of the linear system, (4–5) figure 3, can be written as $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which has eigenvalues $\lambda = [a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}]/2$. The system is stable if both eigenvalues are negative, that is trace $p = a + d < 0$ and determinant $q = ad - bc > 0$ (Drazin, 1992, ch.6). Using the gains of the loops $G_1 = a$, $G_2 = d$ and $G_3 = bc$, then stability can be determined by the sum of the first order loop gains being negative, $G_1 + G_2 < 0$, and their product being bigger than the second order loop gain, $G_1 G_2 > G_3$. That is, there is sufficient dissipation in the system to counteract the effects of the second order loop. One corollary of these conditions is that the system is stable only if: either L_3 is reinforcing and both L_1 and L_2 are balancing; or L_3 is balancing and at least one of the first order loops is also balancing.

The system oscillates $(a+d)^2 < 4(ad-bc)$. This can be expressed in gains as $(G_1-G_2)^2+4G_3 < 0$. Thus $G_3 < 0$ is a necessary condition for oscillation, that is the second order loop L_3 must be balancing.

Classification criteria for stability, saddle behavior, oscillation etc. can be expressed on the the standard $p - q$ plane (Drazin, 1992, p.176) using the loop gains, figure 19.

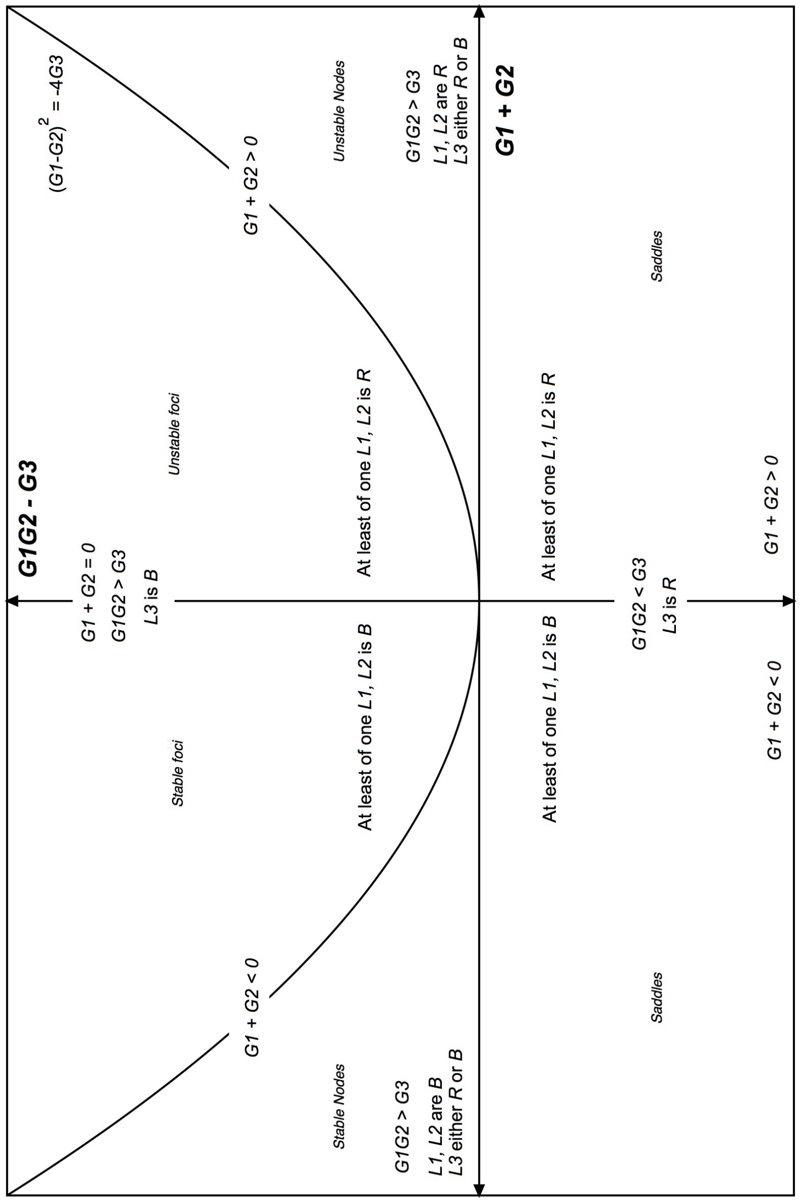

Fig. 19: Stability of second order linear system and loop gain.

Appendix B: Pathway Notation for System Dynamics Models

Causal Pathway Notation

Causal pathways can be indicated by subscripts for the intermediate variables in a causal chain. Let x be a cause of y, write $y = y(x)$, and y be a cause of z, $z = z(y)$. Then

$$z = z(y) = z(y(x)) = z(y(x_{\underline{y}})) = z'(x_{\underline{y}})$$

where the subscript on the argument of the combined function $z' = y \circ z$ indicates the pathway from x to z is via y. Thus in a function combination the name of the intermediary variable becomes a subscript.

For example let $y(x) = 3x^2$ and $z(y) = 2 - 4y$, then $z = 2 - 12x_{\underline{y}}^2$. The subscript makes clear the pathway from x to z via y.

Let $w = w(z)$ be a cause of z then

$$w = w(z(y_{\underline{z}})) = w(z(y(x_{\underline{y}})_{\underline{z}})) = w(z(y(x_{\underline{yz}}))) = w'(x_{\underline{yz}})$$

where $w' = (y \circ z \circ w)$. Thus, in general, for path collections a, b:

$$f(x_{\underline{a}})_{\underline{b}} = f(x_{\underline{ab}})$$

Consider an example with three causal pathways from x to w. Let $y(x) = 3x^2$, $z(x,y) = 2x - 4y$ and $w(x,z) = \sqrt{xz}$. Then

$$w = \sqrt{xz} = \sqrt{x\left(2x_{\underline{z}} - 4y_{\underline{z}}\right)} = \sqrt{x\left(2x_{\underline{z}} - 12x^2_{\underline{zy}}\right)}$$

where the three different pathways are indicated by the different subscripts. The direct pathway from x to w has no subscript.

If such a pathway were part of a system dynamics model then, for numerical simulation, equilibrium analysis etc., the subscripts can be removed and algebraic simplification and numerical computations performed, $w = \sqrt{2x^2(1-6x^2)}$. However for the computation of stock and exogenous impacts, or other network related calculations, the subscripts should be retained.

Stock Impact Notation

Consider a first order system dynamics model with stock x, with net flows f, and with π causal pathways to itself, i.e. π first order loops. Let a_μ be the name for the collection of intermediary auxiliary variables in pathway μ. The system dynamics model in networked equation form is:

$$\dot{x} = f(x) = f(x_{\underline{a_1}}, x_{\underline{a_2}}, \ldots x_{\underline{a_\mu}}, \ldots x_{\underline{a_\pi}}) \tag{23}$$

$x_{\underline{a_\mu}}$ is the variable x along pathway a_μ.

Equation (23) can be written in a more concise form, using the conventions of multivariate calculus and differential geometry, as:

$$\dot{x} = f(x_{\underline{a_\mu}}) \quad \mu = 1 \ldots \pi \tag{24}$$

The stock impacts on x are derived by differentiating (24) by time:

$$\ddot{x} = \frac{\mathrm{d}f}{\mathrm{d}x}\dot{x} = \sum_{\mu=1\ldots\pi} \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right\|_{\underline{a_\mu}} \dot{x}$$

where the pathway derivative is defined by:

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right\|_{\underline{a_\mu}} \triangleq \frac{\partial f}{\partial x_{\underline{a_\mu}}}$$

the derivative along one pathway in the first order model. The stock impacts on x are:

$$\mathrm{I}_{\underline{xa_\mu x}} \triangleq \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right\|_{\underline{a_\mu}}$$

By definition all these stock impacts are first order loop impacts.

Consider an nth order system dynamics model with stocks x_i, $i = 1 \ldots n$, with net flows f_i, and with π_{ij} causal pathways from x_i to x_j. Let a_{ij} be the names of the list of causal pathways from x_i to x_j. a_{ij} is a matrix of lists of possibly differing lengths π_{ij}. An individual causal pathway in the list is indexed by μ_{ij} drawn from the range $1 \ldots \pi_{ij}$, thus giving the matrix of lists $a_{ij\,\mu_{ij}}$, which can be abbreviated to $a_{ij\,\mu}$ without confusion. Each element of each list is a collection of intermediary auxiliary variables in pathway μ_{ij}. The nth order system dynamics model in a concise networked equation form is:

$$\dot{x}_i \quad = \quad f_i(x_{j\underline{a_{ji\,\mu}}}) \quad i,j = 1 \ldots n; \;\; \mu_{ji} = 1 \ldots \pi_{jj} \tag{25}$$

$x_{j\underline{a_{ji\,\mu}}}$ is the variable x_j along pathway $a_{ji\,\mu} \equiv a_{ji\,\mu_{ji}}$ connected to x_i. There are π_{ji} pathways connecting these variables.

The stock impacts on x_i are derived by differentiating (25) by time:

$$\ddot{x}_i = \sum_{j=1\ldots n} \frac{\partial f_i}{\partial x_j} \frac{\dot{x}_j}{\dot{x}_i} \dot{x}_i = \sum_{j=1\ldots n} \; \sum_{\mu_{ji}=1\ldots\pi_{ji}} \frac{\partial f_i}{\partial x_j}\bigg\|_{\underline{a_{ji\,\mu}}} \frac{\dot{x}_j}{\dot{x}_i} \dot{x}_i \tag{26}$$

where the pathway derivative is defined by:

$$\frac{\partial f_i}{\partial x_j}\bigg\|_{\underline{a_{ji\,\mu}}} \triangleq \frac{\partial f}{\partial x_{j\,\underline{a_{ji\,\mu}}}} \tag{27}$$

the derivative along one pathway $a_{ji\,\mu}$ in the nth order model. Note in $x_{j\,\underline{a_{ji\,\mu}}}$ the underlined subscripts $\underline{a_{ji\,\mu}}$ themselves subscripted, define the pathway, whereas the non-underlined subscript j indicates the variable name. The use of the underline subscript notation should avoid confusion for the many other uses of subscripts in dynamical models.

The stock impacts of x_j on x_i are:

$$\mathrm{I}_{\underline{x_j a_{ji\,\mu} x_i}} \triangleq \frac{\partial f_i}{\partial x_j}\bigg\|_{\underline{a_{ji\,\mu}}} \frac{\dot{x}_j}{\dot{x}_i}$$

When $i = j$ these are first order loop impacts. When $i \neq j$ these impacts may be part of higher order loops, however this is not guaranteed, as some stocks may not be part of feedback loops. Thus the term stock impact is preferred. If impacts are part of higher order loops they may be referred to as loop impacts, but as in the case of the economic long-wave model, they may be part of more than one loop.

The notation is easily extended to include multiple pathways from any number of exogenous forces.

The double vertical line notation for the pathway derivative has been chosen for two reasons. Firstly to avoid confusion with the use of the single line in the evaluation of integrals. Secondly to avoid confusion with different index notations for differentiation.

For example, using the comma notation for partial differentiation, $f_{i,j}$, the pathway derivative (27) can be written:

$$f_{i,j}\|_{\underline{a_{ji\,\mu}}} \triangleq f_{i,j\|\underline{a_{ji\,\mu}}}$$

the partial derivative of variable i by variable j along the pathway $\underline{a_{ji\,\mu}}$ from j to i.

In differential geometry covariant derivatives are used as a coordinate free form of differentiation, with semicolons and single vertical lines as notation (Rund, 2012). Thus a covariant derivative along a pathway would have no confusion between the derivative notation and the pathway notation. For example $f_{i|j\|\underline{a_{ji\,\mu}}}$ is the horizontal covariant derivative of variable i by variable j along pathway $\underline{a_{ji\,\mu}}$. Although there is no immediate use for this notation it is noted that equations (26) may be geodesics in a curved space for certain system dynamics models, suggesting a geometrical approach could be useful as a future line of research in this area.